1.Integral Relation

1.1 Relation between integral of the function and integral of its inverse-function

Theorem

Suppose $y = f(x)$ is a function defined in xy plane, differential with respect to x, and not a constant function, then,

$$\int ydx + \int xdy = xy + c \quad (c\text{:integral constant})$$

(proof)

$$(xy)' = y + x \cdot \frac{dy}{dx}$$

$$xy = \int ydx + \int x \cdot \frac{dy}{dx} dx + c$$

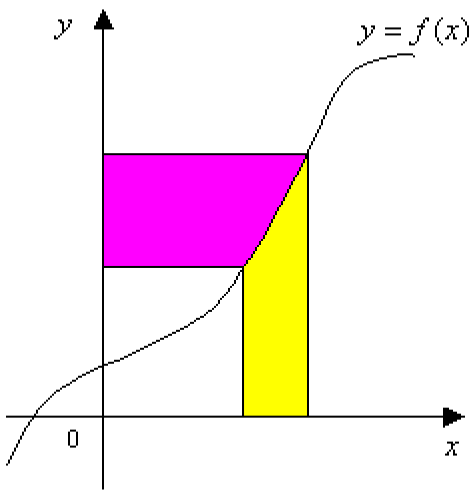
$$= \int ydx + \int xdy + c$$

(end of proof)

Now, $\int ydx$ and $\int xdy$ are both indefinite integrals and it seems to have no relation to their intervals of integration. Their description within the same expression means

$f : dx \to dy$,

the interval of integration, as shown in the right figure, thus corresponding to the function f.



Therefore, the relation will readily be understood using the following expression:

$$\int_{x_0}^{x} ydx + \int_{y_0}^{y} xdy = xy - x_0 y_0$$

where $c = -x_0 y_0$.

However, as is shown by the proof above,

$$\int ydx + \int xdy = xy + c$$

can be understood as an application of integration by parts.

Exemplary calculations of relation

Note: Integration constant is to be abbreviated below.

Example 1:

For $y=\sin^{-1}x$, its inverse function is $x=\sin y$ and

$$\int \sin^{-1}xdx+\int \sin ydy = x\sin^{-1}x$$

$$\int \sin^{-1}xdx = x\sin^{-1}x+\cos y$$

$$\cos y = \cos(\sin^{-1}x)$$

$$=\cos(\cos^{-1}\sqrt{1-x^2})$$

$$=\sqrt{1-x^2}$$

$$\therefore \int \sin^{-1}xdx = x\sin^{-1}x+\sqrt{1-x^2}$$

Example 2:

For $y=\log x$, its inverse function is $x=e^y$ and

$$\int \log xdx+\int e^y dy = x\log x$$

$$\int \log xdx = x\log x - e^y$$

$$= x\log x - x$$

Example 3:

For $y=x^3+x^2+x+1$, set its inverse function as $x=g(y)$,

$$\int g(y)dy = x(x^3+x^2+x+1)-\int (x^3+x^2+x+1)dx$$

$$= x^4+x^3+x^2+x-(\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x)$$

$$=\frac{3x^4}{4}+\frac{2x^3}{3}+\frac{x^2}{2}$$

Example 4:

For $y=\frac{1}{x}$,

the left side $=\int ydx+\int xdy$

$$=\log x+\log y$$

$$= \log x + \log \frac{1}{x}$$

$$= 0$$

and the right side $= 1 + c$

Equation between both sides does not seem to become valid, but essentially,

the left side $= [xy]_{x_0}^{x}$

$$= xy - x_0 y_0$$

$$= xy - 1$$

and c equals to "the constant -1".

1.2 Relation for nondifferential function

Theorem

Suppose $y = f(x)$ is a function defined in xy plane, and continuous and bounded variation with respect to the closed interval $[a,b]$,

$$\int_a^b f(x)dx + \int_a^b xdf(x) = bf(b) - af(a)$$

where $\int$ means Lebesgue integral.

This theorem can be established by application of Lebesgue integral.

Using the figure below to help our understanding, it might be possible to find out that whatever function will do only if that function is continuous and has no infinite-point for the closed interval $[a,b]$.

Therefore,

$\int ydx + \int xdy = xy + c$ for 1, 1

can also be understood as a special case of Lebesgue integral.