2.1 Property of Laurent expansion

Theorem

Suppose $w = f(z)$ is a function in complex plane and regular for the region

$$D : 0 \leqq R_1 < |z-\alpha| < R_2 \leqq \infty ;$$

and $w = f(z)$ is expansive by Laurent expansion and $f(z) = \sum_{\nu=-\infty}^{\infty} c_\nu (z-\alpha)^\nu$.

Here, set $z = \alpha + re^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$ and write $f(z) = g(r,\theta)$ in polar form: then

$g(r,\theta) = g(r,\theta + 2\pi/n)$ constantly for $g(r,\theta) \Leftrightarrow f(z) = \sum_{\nu=-\infty}^{\infty} c_{n\nu} (z-\alpha)^{n\nu}$

where n is a natural number.

(proof)

$(\Leftarrow)$

$$f(z) = \sum_{\nu=-\infty}^{\infty} c_{n\nu} (z-\alpha)^{n\nu}$$

$$= \sum_{\nu=-\infty}^{\infty} c_{n\nu} r^{n\nu} e^{in\nu\theta}$$

now, for $^{\forall}\nu$

$$\begin{aligned} e^{in\nu(\theta + 2\pi/n)} &= \cos(n\nu\theta + 2\pi\nu) + i\sin(n\nu\theta + 2\pi\nu) \\ &= \cos(n\nu\theta) + i\sin(n\nu\theta) \\ &= e^{in\nu\theta} \end{aligned}$$

$$\therefore g(r,\theta) = g(r,\theta + 2\pi/n)$$

$(\Rightarrow)$

set $f(z) = g(r,\theta) = \sum_{\nu=-\infty}^{\infty} c_\nu r^\nu e^{i\nu\theta}$: then

$$g(r,\theta + 2\pi/n) = \sum_{\nu=-\infty}^{\infty} c_\nu r^\nu e^{i\nu(\theta + 2\pi/n)} .$$

From $g(r,\theta) = g(r,\theta + 2\pi/n)$ we can establish the expression for ${}^{\forall}\nu$:

$$c_\nu r^\nu e^{i\nu\theta} = c_\nu r^\nu e^{i\nu(\theta+2\pi/n)} .$$

$r \neq 0$ allows to be valid $c_\nu e^{i\nu\theta} = c_\nu e^{i\nu(\theta+2\pi/n)}$.

Assume $c_\nu \neq 0$,

$$e^{i\nu\theta} = e^{i\nu(\theta+2\pi/n)}$$

$$= e^{i\nu\theta} \cdot e^{2\pi i\nu/n}$$

$$\therefore e^{2\pi i\nu/n} = 1$$

Unless ν is a multiple of n, the equation above becomes contradictory, thus leading to $c_\nu = 0$.

Now, suppose ν is an integer, we can establish the expression:

$$f(z) = \sum_{\nu=-\infty}^{\infty} c_{n\nu} (z-\alpha)^{n\nu} .$$

(end of proof)

The following result is clear, but once more I would like to set out for you reference:

1. Suppose $w = f(z)$ is a function in complex plane: then the necessary condition where it is expansive using Laurent expansion for $D: 0 \leqq R_1 < |z-\alpha| < R_2 \leqq \infty$ is

   $g(r,\theta) = g(r,\theta + 2\pi)$ constantly for $g(r,\theta)$.
2. Suppose $w = f(z)$ is a function in complex plane and expansive by Laurent expansion at neighbourhood of the point α .

If $g(r,\theta) = g(r,\theta + 2\pi/n)$ constantly for $g(r,\theta)$ where n is a natural number more than 1, then

ν -th derived function $f^{(\nu)}(\alpha) = 0$ where ν ($\nu > 0$)is not a multiple of n .

From 2, $f^{(n)}(\alpha)$ now “may be known from the periodicity in terms of its polar-form function with center at α" instead of formerly "can be known upon substituting α into it after differentiating it n times."

2.2 Periodicity of funtion

Suppose $w = f(z)$ is a entire function defined in complex plane.

$f(z)$ is transformed to its polar form $f(z) = g_\alpha(r,\theta)$ using $z = \alpha + re^{i\theta}$ at a any point α.

Set n as a natural number not less than 2, then the number of α for $g_\alpha(r,\theta)$ such that constantly satisfies $g_\alpha(r,\theta) = g_\alpha(r,\theta + 2\pi/n)$ in complex plane is

1. null;
2. only one; or
3. infinite (countable) at regular intervals on a line,

   each point satisfying the equation $g_\alpha(r,\theta) = g_\alpha(r,\theta + \pi)$.

(proof)

1.Suppose a and b are real-number constants and $w = az + b$, they clearly satisfies 1.

2.Suppose n is a natural number more than 1 and $w = z^n$, they clearly satisfies 2..

3.Let us consider on a condition: $w = \cos z$; $\alpha = n\pi$ $(n = \cdots -2,-1,0,1,2,\cdots)$.

$$\cos(n\pi + z) = \frac{1}{2}\left(e^{i(n\pi+z)} + e^{-i(n\pi+z)}\right)$$

$$= \frac{1}{2}\left(e^{in\pi}e^{iz} + e^{-in\pi}e^{-iz}\right)$$

$$= \begin{cases} \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cos z & (\text{n : an even number}) \\ \frac{1}{2}\left(-e^{iz} - e^{-iz}\right) = -\cos z & (\text{n : an odd number}) \end{cases}$$

Therefore, the periodicity of $w = \cos z$ is to be determined the region: $0 \leqq \Re(z) < \pi$.

For $\alpha = 0$,

$$g_0(r,\theta + \pi) = \frac{1}{2}\left(e^{ir(\cos(\theta+\pi)+i\sin(\theta+\pi))} + e^{-ir(\cos(\theta+\pi)+i\sin(\theta+\pi))}\right)$$

$$= \frac{1}{2}\left(e^{-ir(\cos\theta+i\sin\theta)} + e^{ir(\cos\theta+i\sin\theta)}\right)$$

$$= g_0(r,\theta)$$

Therefore, $g_\alpha(r,\theta)$ has periodicity at α=0.

For n as an even number,

$$g_{n\pi}(r,\theta+\pi)=\frac{1}{2}\left(e^{i(n\pi+r(\cos(\theta+\pi)+i\sin(\theta+\pi)))}+e^{-i^{i(n\pi+r(\cos(\theta+\pi)+i\sin(\theta+\pi)))}}\right)$$

$$=\frac{1}{2}\left(e^{i(n\pi-r(\cos\theta+i\sin\theta))}+e^{-i(n\pi-r(\cos\theta+i\sin\theta))}\right)$$

$$=\frac{1}{2}\left(e^{-ir(\cos\theta+i\sin\theta)}+e^{ir(\cos\theta+i\sin\theta)}\right)$$

$$=\frac{1}{2}\left(e^{-in\pi-ir(\cos\theta+i\sin\theta)}+e^{in\pi+ir(\cos\theta+i\sin\theta)}\right)$$

$$=g_{n\pi}(r,\theta)$$

For n as an odd number,

$$g_{n\pi}(r,\theta+\pi)=\frac{1}{2}\left(e^{i(n\pi+r(\cos(\theta+\pi)+i\sin(\theta+\pi)))}+e^{-i^{i(n\pi+r(\cos(\theta+\pi)+i\sin(\theta+\pi)))}}\right)$$

$$=\frac{1}{2}\left(e^{i(n\pi-r(\cos\theta+i\sin\theta))}+e^{-i(n\pi-r(\cos\theta+i\sin\theta))}\right)$$

$$=\frac{1}{2}\left(e^{-ir(\cos\theta+i\sin\theta)}+e^{ir(\cos\theta+i\sin\theta)}\right)$$

$$=\frac{1}{2}\left(e^{-in\pi-ir(\cos\theta+i\sin\theta)}+e^{in\pi+ir(\cos\theta+i\sin\theta)}\right)$$

$$=g_{n\pi}(r,\theta)$$

Therefore, $g_\alpha(r,\theta)=g_\alpha(r,\theta+\pi)$ is valid at a point $\alpha=n\pi$ on a line (real-number axis). Thus, there is a function satisfying 3.

In this case, it is unknown whether or not on real-number axis is the point $\beta(0<\beta<\pi)$, it is also unknown whether or not any point on any other axis than real-number one has periodicity, which will be illustrated later in the property of general entire function.

Let us consider using general entire function.

First, suppose there are 2 points having periodicity;

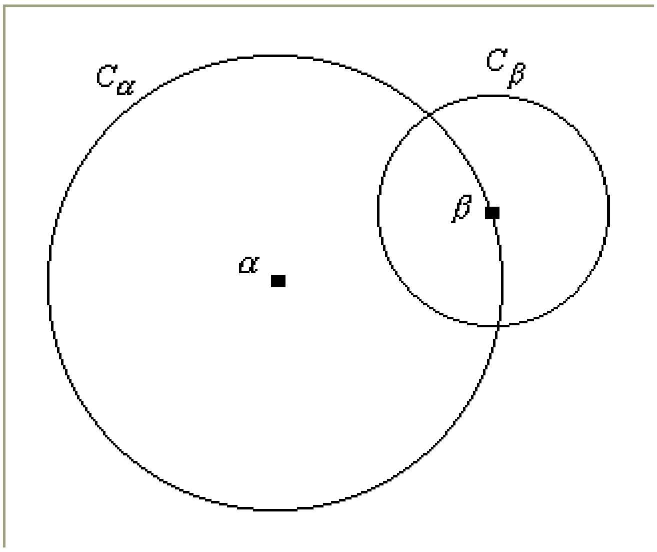
Assume $g_\alpha(r,\theta)=g_\alpha(r,\theta+2\pi/m)$ at $z=\alpha$; and

$g_\beta(r,\theta)=g_\beta(r,\theta+2\pi/n)$ at $z=\beta$.

Put $m > 2$、$n > 2$.

Suppose there is a point β on a circle C_α with center at α.

Assume another circle C_β with center at β having a any radius.



Suppose a point β_1 is generated by revolving a point β counterclockwise by $2\pi/m$,

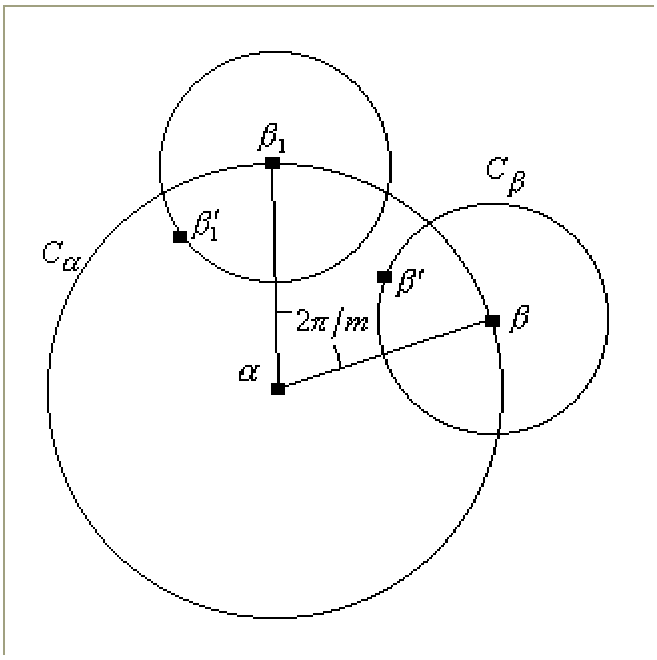
from $g_\alpha(r,\theta) = g_\alpha(r,\theta + 2\pi/m)$ we can establish

$f(\beta) = f(\beta_1)$.

Likewise, suppose a point β_1' is generated by revolving a point β' counterclockwise by $2\pi/m$,

from $g_\alpha(r,\theta) = g_\alpha(r,\theta + 2\pi/m)$ we can establish

$f(\beta') = f(\beta_1')$.



Therefore, the property allowing to be valid at $z = \beta$

$$g_{\beta}(r,\theta) = g_{\beta}(r,\theta + 2\pi/n)$$

can also be applied to β_1, giving

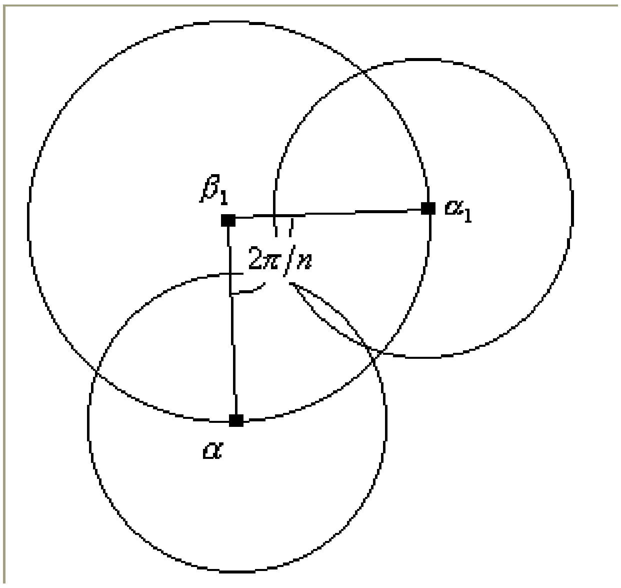
$$g_{\beta 1}(r,\theta) = g_{\beta 1}(r,\theta + 2\pi/n)$$

The number of periodic points is 3.

For the foregoing, we began with α ; Now, beginning with β_1 to generate α_1 from α leads to

$$g_{\alpha 1}(r,\theta) = g_{\alpha 1}(r,\theta + 2\pi/m).$$

Therefore, the number of periodic points is 4.



Repeating this will cause a lattice point having periodicity to be infinite (countable).

Suppose either m or n is not less than 3 and the other is 2.
This condition, like that of $m > 2$、$n > 2$, will also cause a lattice point having periodicity to be infinite (countable).

Suppose periodic points is infinite (countable) in the whole complex plane like a lattice, $w = f(z)$ has a finite value at $|z| = \infty$.

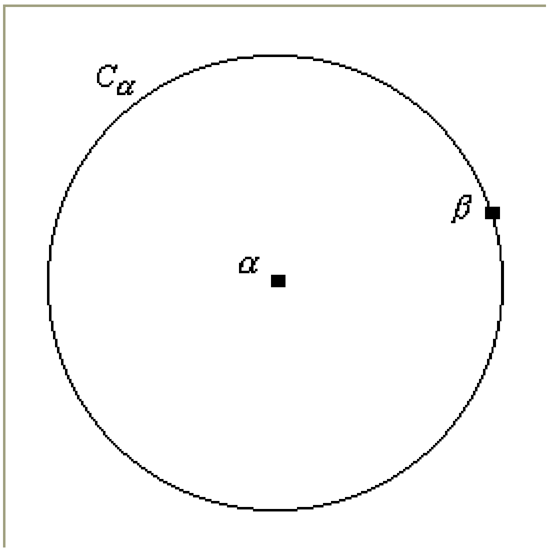
From Liouville's Theorem, $w = f(z)$ must be a constant.

Therefore, it is contradictory that it has periodicity at infinite (countable) point in the whole complex plane.

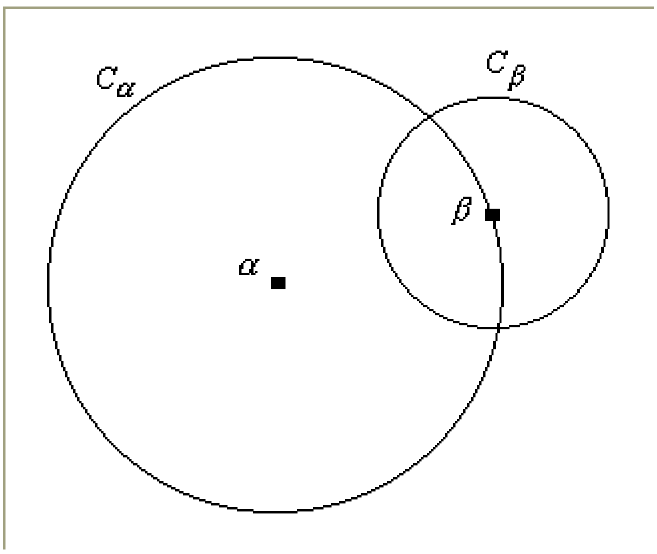
Naturally, it is also contradictory that it has periodicity in continuous potency in the whole complex plane.

Put $m = n = 2$.

Suppose there is a point β on a circle C_α with center at α.



Assume another circle C_β with center at β having a any radius.



Suppose a point β_1 is generated by revolving a point β counterclockwise by π ,

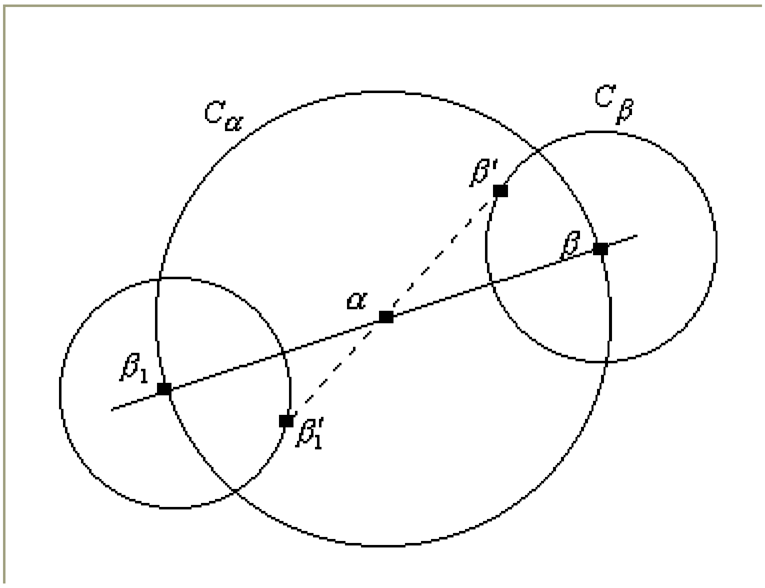
from $g_\alpha(r,\theta) = g_\alpha(r,\theta+\pi)$ we can establish

$f(\beta) = f(\beta_1)$.

Likewise, suppose a point β_1' is generated by revolving a point β' counterclockwise by π ,

from $g_\alpha(r,\theta) = g_\alpha(r,\theta+\pi)$ we can establish

$f(\beta') = f(\beta_1')$.



Therefore, the property allowing to be valid at $z = \beta$

$g_\beta(r,\theta) = g_\beta(r,\theta+\pi)$

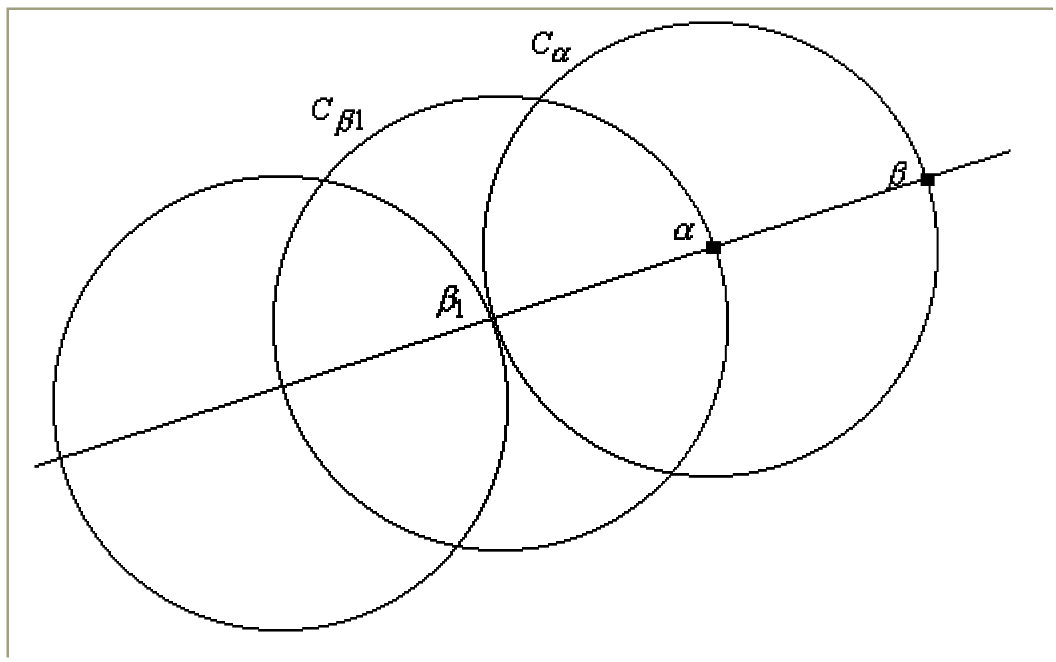
can be also applied to β_1, giving

$g_{\beta 1}(r,\theta) = g_{\beta 1}(r,\theta+\pi)$.

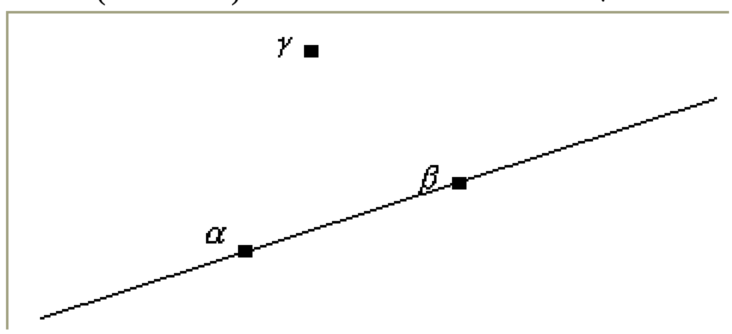
Repeating this will cause a point α such that satisfies

$g_\alpha(r,\theta) = g_\alpha(r,\theta+\pi)$

to be infinite (countable) on a line at regular intervals.

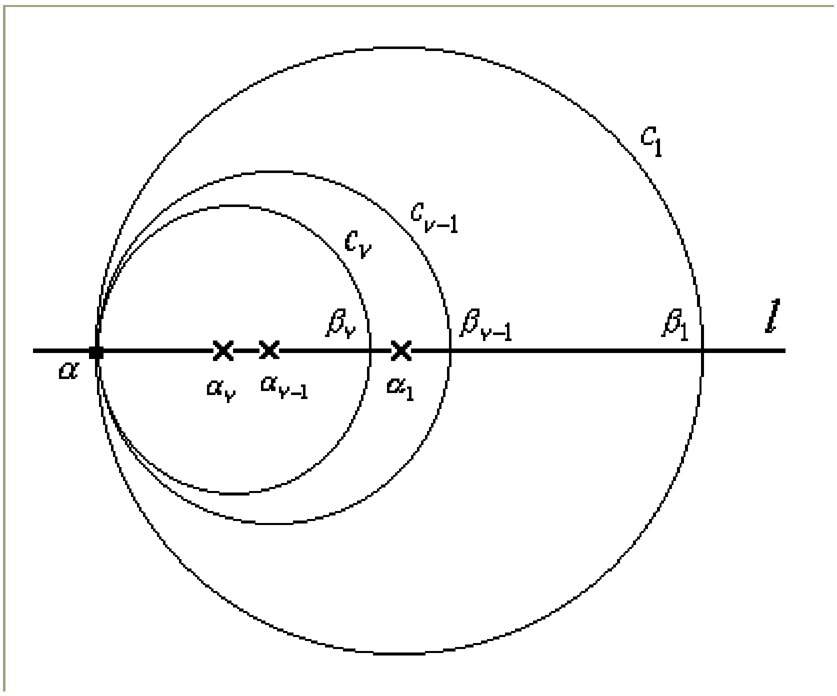


For $m = n = 2$, if $g_\gamma(r,\theta)$ has periodicity at point γ not on the line between points α and β as shown in the figure below, the number of lattice points having periodicity will be infinite (countable) as with the case for $m > 2$、$n > 2$.



Therefore, it is contradictory.

Suppose $m = n = 2$ and there is a periodic point in continuous potency on a line.



On the line l is a sequence of points $\{\alpha_v\}$ such that converges into a point α .

Suppose $g_{\alpha v}(r,\theta) = g_{\alpha v}(r,\theta+\pi)$ is valid at each point α_v .

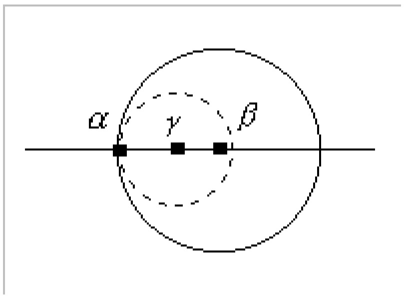
Assume a circle c_v such that has center at α_v as well as α on itself.

Assume β_v that is the other point of intersection than α between the lines l and c_v . From the periodicity at α_v we can establish

$f(\alpha) = f(\beta_v)$.

Since a sequence of points $\{\beta_v\}$ converges into a point α and $f(\alpha) = f(\beta_v)$ is valid for a any v , Unicity Theorem causes $f(z)$ to constantly be a constant $f(\alpha)$ at neighbourhood of the point α .
Therefore, it is contradictory.

Supposing on the line with connecting the points α, β, the point γ between the line α and β satisfies $g_\gamma(r,\theta) = g_\gamma(r,\theta+\pi)$.



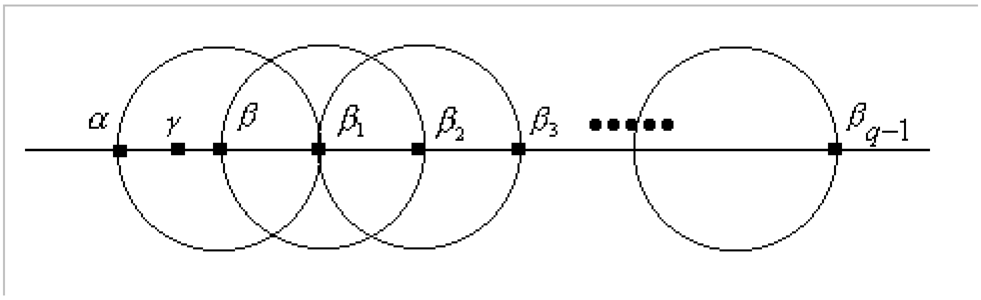
Let the length of segment of $\alpha\beta$ be $|\alpha\beta|$ and the length of segment of $\alpha\gamma$ be $|\alpha\gamma|$.

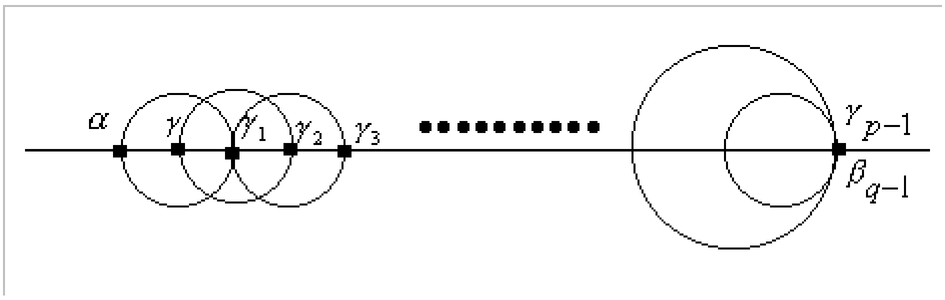
Let the value of ratio of $|\alpha\beta|$ and $|\alpha\gamma|$ be rational numbers, and each of p, q be relatively prime numbers.

$$\frac{|\alpha\gamma|}{|\alpha\beta|} = \frac{q}{p}$$

From $g_\beta(r,\theta) = g_\beta(r,\theta+\pi)$, $g_\gamma(r,\theta) = g_\gamma(r,\theta+\pi)$,

we can establish the result which will be as the illustrations below.



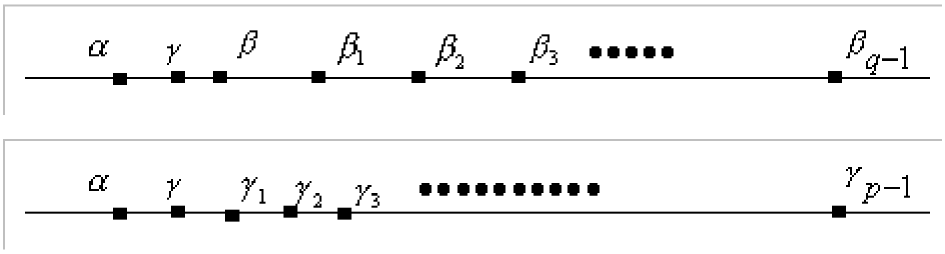


Thus the sequence of points $\{\beta_j\},\{\gamma_i\}$ will be formed and β_{q-1},γ_{p-1} will be an identical point.
In case of the sequence of points $\{\beta_j\}$, $\{\gamma_i\}$, the length of the points between some β_{j0} and

γ_{i0} is shortest.

New periodicity of funtion can be developed from β_{j0} and γ_{i0}.

When focused on only the sequence of points;



The sequence of points of a segment of line $\alpha\beta_{q-1}$ as divided by q, and divided by p, which will be equivalent to determine the sequence of points to include each sequence.
As p and q are the relatively prime numbers, the segment of line $\alpha\beta_{q-1}$ can be divided by

pq which will be a division to satisfy the condition.

Therefore, in case of the ratio of $|\alpha\beta|$ and $|\alpha\gamma|$ is a rational number, and with p , q

be relatively prime numbers,

$$\frac{|\alpha\gamma|}{|\alpha\beta|} = \frac{q}{p}$$

the segment of line $\alpha\gamma$ divided by q will be the most closest periodic point to α; and with this point as the criteria, we can establish the points which have a periodicity to be infinite (countable).

Let the value of the ratio of $|\alpha\beta|$ and $|\alpha\gamma|$ be irrational number.

From $g_\beta(r,\theta) = g_\beta(r,\theta+\pi)$, $g_\gamma(r,\theta) = g_\gamma(r,\theta+\pi)$,

with the expansion of α from this, the sequence of points $\{\beta_j\}(-\infty < j < \infty)$, $\{\gamma_i\}(-\infty < i < \infty)$

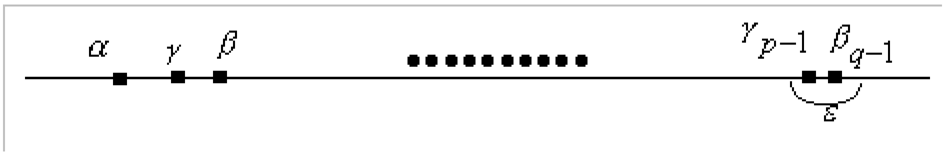
can be created; however, no coincident will be found in $\{\beta_j\}\{\gamma_i\}$.

Any ε is considered.
And some p, q are considered the relatively prime numbers.,

$$\left| \frac{|\alpha\gamma|}{|\alpha\beta|} - \frac{q}{p} \right| < \varepsilon$$

The value p, q to satisfy the above equation exists.

In the point row $\{\beta_j\}$、$\{\gamma_i\}$, the length between β_{q-1}、γ_{p-1} will be less than ε .



$\beta_{q-1}, \gamma_{p-1}$ as the criterion, the development of the periodicity of funtion can be possible.

In this development, a distant of less than ε from the point α can be created around the point α , and this will enable to create the sequence of points convergent to the point α .

The ratio of $|\alpha\beta|$ and $|\alpha\gamma|$ are impossible to be irrational numbers.

Therefore, and a entire function to satisfy 1,2 or 3 exists. Also, a entire function is periodic, it must be limited to the conditions 2 or 3.