2.3 Relation between the regular and principal parts of the Laurent expansion

Let $w = f(z)$ be a function in the complex plane. The function is regular in domain $D: 0 \leqq R_1 < |z-\alpha| < R_2 \leqq \infty$. Suppose that $w = f(z)$ can be expanded into a Laurent expansion.

Let C_1 and C_2 be circles with radii r_1 and r_2, with $R_1 < r_1 < |z-\alpha| < r_2 < R_2$.

We have

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n (z-\alpha)^n + \sum_{\nu=1}^{\infty} \frac{a_{-n}}{(z-\alpha)^n},$$

with $a_0 = \frac{1}{2\pi i}\int_{C2} \frac{f(z)}{z-\alpha} dz$.

For $n = 1, 2, \ldots$,

$$a_n = \frac{1}{2\pi i}\int_{C2} \frac{f(z)}{(z-\alpha)^{n+1}} dz$$

$$a_{-n} = \frac{1}{2\pi i}\int_{C1} (z-\alpha)^{n-1} f(z) dz.$$

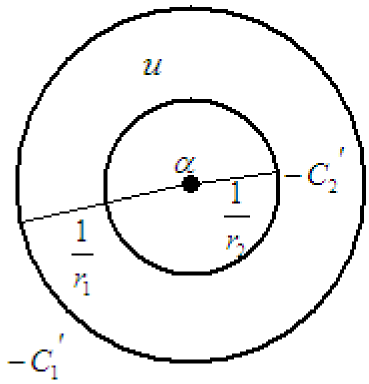
Using the variable transformation

$$z - \alpha = \frac{1}{u-\alpha},$$

$$w = f(z)$$

$$= f\left(\alpha + \frac{1}{u-\alpha}\right).$$

$$= g(u)$$



Let us consider the following expressions:

$$C_1' = \left\{ u : u - \alpha = \frac{1}{z-\alpha}, z \in C_1 \right\}$$

$$C_2' = \left\{ u : u - \alpha = \frac{1}{z-\alpha}, z \in C_2 \right\}.$$

Since C_1' and C_2' are both clockwise, the counterclockwise circles are represented as $-C_1'$ and $-C_2'$.

$w = g(u)$ is regular in domain $D': 0 \leqq \frac{1}{R_2} < |u-\alpha| < \frac{1}{R_1} \leqq \infty$ and

$$g(u) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{(u-\alpha)^n} + \sum_{\nu=1}^{\infty} a_{-n} (u-\alpha)^n, \text{ with}$$

$$a_0 = \frac{1}{2\pi i}\int_{-C_2'} \frac{g(u)}{u-\alpha}\,du = \frac{1}{2\pi i}\int_{-C_1'} \frac{g(u)}{u-\alpha}\,du\,.$$

For $n = 1,2,\ldots$,

$$a_{-n} = \frac{1}{2\pi i}\int_{C_1'} \frac{g(u)}{(u-\alpha)^{n+1}}\,du$$

$$a_n = \frac{1}{2\pi i}\int_{C_2'} (u-\alpha)^{n-1} g(u)\,du\,.$$

Proof:

It is trivial to show that $g(u)$ is regular in domain D' : $0 \leqq \frac{1}{R_2} < |u-\alpha| < \frac{1}{R_1} \leqq \infty$.

Transforming $f(z) = a_0 + \sum_{n=1}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-\alpha)^n}$

by using the variable transformation

$$z - \alpha = \frac{1}{u-\alpha},$$

we obtain

$$g(u) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{(u-\alpha)^n} + \sum_{\nu=1}^{\infty} a_{-n} (u-\alpha)^n .$$

Considering that

$$C_1 = \{z : z = \alpha + r_1 e^{i\theta} (0 \leqq \theta \leqq 2\pi)\}$$
$$C_2 = \{z : z = \alpha + r_2 e^{i\theta} (0 \leqq \theta \leqq 2\pi)\},$$

we have

$$-C_1{}' = \left\{u : u = \alpha + \frac{1}{r_1} e^{-i\theta} (0 \leqq \theta \leqq 2\pi)\right\}$$

$$-C_2{}' = \left\{u : u = \alpha + \frac{1}{r_2} e^{-i\theta} (0 \leqq \theta \leqq 2\pi)\right\}.$$

Therefore,

$$a_0 = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-\alpha} dz$$
$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\alpha + r_2 e^{i\theta})}{r_2 e^{i\theta}} \frac{dz}{d\theta} d\theta$$
$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\alpha + r_2 e^{i\theta})}{r_2 e^{i\theta}} i r_2 e^{i\theta} d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + r_2 e^{i\theta}) d\theta .$$

$$\frac{1}{2\pi i} \int_{-C_2{}'} \frac{g(u)}{u-\alpha} du$$

$$= \frac{1}{2\pi i}\int_{2\pi}^{0} \frac{g\left(\alpha + \frac{1}{r_2}e^{-i\theta}\right)}{\frac{1}{r_2}e^{-i\theta}} \frac{du}{d\theta} d\theta$$

$$= \frac{1}{2\pi i}\int_{0}^{2\pi} \frac{f(\alpha + r_2 e^{i\theta})}{\frac{1}{r_2}e^{-i\theta}} \frac{i}{r_2} e^{-i\theta} d\theta$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} f(\alpha + r_2 e^{i\theta}) d\theta .$$

$$\therefore a_0 = \frac{1}{2\pi i}\int_{-C_2{}'} \frac{g(u)}{u-\alpha} du .$$

Since $w = g(u)$ is regular in domain $D' : 0 \leqq \frac{1}{R_2} < |u-\alpha| < \frac{1}{R_1} \leqq \infty$,

$$a_0 = \frac{1}{2\pi i}\int_{-C_1{}'} \frac{g(u)}{u-\alpha} du .$$

For

$n = 1,2,\ldots$, we have

$$a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-\alpha)^{n+1}} dz$$

$$= \frac{1}{2\pi i}\int_{0}^{2\pi} \frac{f(\alpha + r_2 e^{i\theta})}{\left(r_2 e^{i\theta}\right)^{n+1}} \frac{dz}{d\theta} d\theta$$

$$= \frac{1}{2\pi i}\int_{0}^{2\pi} \frac{f(\alpha + r_2 e^{i\theta})}{\left(r_2 e^{i\theta}\right)^{n+1}} i r_2 e^{i\theta} d\theta$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} \frac{f(\alpha + r_2 e^{i\theta})}{\left(r_2 e^{i\theta}\right)^{n}} d\theta .$$

$$\frac{1}{2\pi i}\int_{C_2{}'} (u-\alpha)^{n-1} g(u) du$$

$$= \frac{1}{2\pi i}\int_{2\pi}^{0} \left(\frac{1}{r_2}e^{-i\theta}\right)^{n-1} g\left(\alpha + \frac{1}{r_2}e^{-i\theta}\right) \frac{du}{d\theta} d\theta$$

$$= \frac{1}{2\pi i}\int_{0}^{2\pi} \left(\frac{1}{r_2}e^{-i\theta}\right)^{n-1} f(\alpha + r_2 e^{i\theta}) \frac{i}{r_2} e^{-i\theta} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\alpha + r_2 e^{i\theta})}{\left(r_2 e^{i\theta}\right)^n} d\theta .$$

$$\therefore a_n = \frac{1}{2\pi i} \int_{C_2'} (u-\alpha)^{n-1} g(u) du .$$

Using a similar calculation, we obtain

$$a_{-n} = \frac{1}{2\pi i} \int_{C_1'} \frac{g(u)}{(u-\alpha)^{n+1}} du .$$

End of the proof

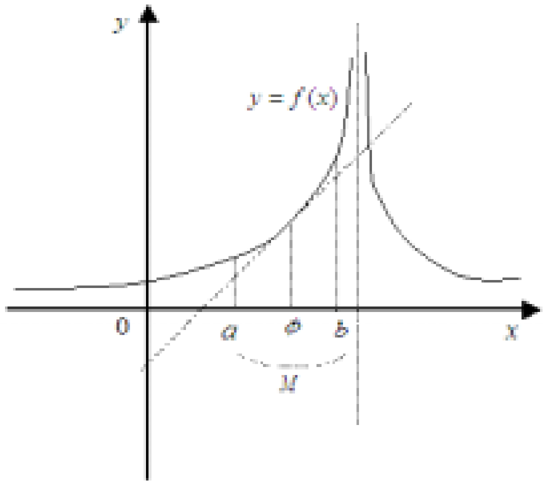
Therefore, the coefficient of the regular part of $f(z)$ coincides with the coefficient of the principal part of $g(u)$ after variable transformation, and the coefficient of the principal part of $f(z)$ coincides with the coefficient of the regular part of $g(u)$. The coefficient a_{-n} of the principal part of $f(z)$ corresponds to the n-th differential coefficient of $g(u)$.

Generalization of the Taylor Formula

[Mean value theorem]

If $f(x)$ is continuous in $[a,b]$ and differentiable in (a,b), there exists ϕ such that

$$\frac{f(b)-f(a)}{b-a} = f'(\phi), a<\phi<b$$.



[Taylor formula]

Let $f(x)$ be n-times differentiable in the interval $[a,b]$.

If $a<x<b$, then

$$f(x) = f(a)+(x-a)\frac{f'(a)}{1!}+(x-a)^2\frac{f^{(2)}(a)}{2!}+\cdots+(x-a)^{n-1}\frac{f^{(n-1)}(a)}{(n-1)!}+(x-a)^n\frac{f^{(n)}(\phi)}{n!}$$.

Here, $a<\phi<b$.

Moreover, suppose that the convergence radius of x is

$|x-a|<M$.

[Generalization]

First, let us consider a to be a constant.

Let us apply the variable transformation $x = a + \frac{1}{X - a}$.

$$f(x) = f\left(a + \frac{1}{X - a}\right)$$

$$= F(X).$$

This is equivalent to representing the inverse of x centered on a as X, replacing the function $f(x)$ by $F(X)$.

Here, we will generalize the mean value theorem.

In the equation $\frac{f(b) - f(a)}{b - a} = f'(\phi)$, let us consider

$$a = \lim_{A \to \infty}\left(a + \frac{1}{A - a}\right),$$

$$b = a + \frac{1}{B - a},$$

$$\phi = a + \frac{1}{\Phi - a}.$$

We have

$$b - a = \frac{1}{B - a} - \lim_{A \to \infty}\frac{1}{A - a}$$

$$= \frac{1}{B - a}.$$

The mean value theorem can be transformed as follows:

$$\frac{F(B) - f(a)}{\frac{1}{B - a}} = f'\left(a + \frac{1}{\Phi - a}\right)$$

$$F(B) = f(a) + \frac{f'\left(a + \frac{1}{\Phi - a}\right)}{B - a},$$

where $B < \Phi < \infty$.

Substituting the above expression into Taylor's formula, we have

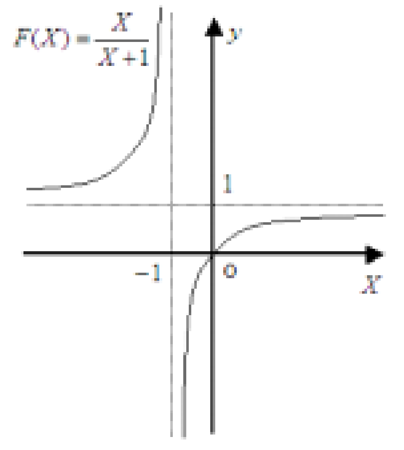
$$F(X) = f(a) + \frac{f'(a)}{1!(X-a)} + \frac{f^{(2)}(a)}{2!(X-a)^2} + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!(X-a)^{n-1}} + \frac{f^{(n)}\left(a + \frac{1}{\Phi - a}\right)}{n!(X-a)^n},$$

where

$B < \Phi < \infty$ [A1] .

The convergence radius is

$$|X-a| > \frac{1}{M}.$$



Calculation example:

$$F(X) = \frac{X}{X+1}$$

and $a = 0$.

Assume that $1 < B < X$.

Performing the variable transformation, we have

$$f(x) = \frac{1}{x+1}.$$

Since the Taylor expansion of the above function is possible in $0 \leqq x < 1$,

we have

$$f(x) = 1 - x + x^2 - x^3 + \cdots.$$

Therefore, the Taylor expansion of $F(X)$ in the range $1 < X$ is

$$F(X) = 1 - \frac{1}{X} + \frac{1}{X^2} - \frac{1}{X^3} + \cdots.$$