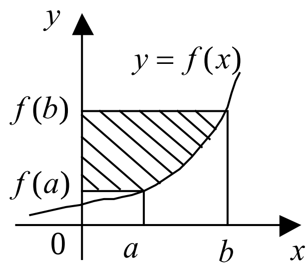
Let $y = f(x)$ be a continuous, monotonic function in $[a,b]$. Moreover, consider that $f'(x)$ exists and is continuous in that interval.

(1) $\int_a^b xf'(x)dx$ corresponds to the shaded area in the figure on the right.



(Proof)

Integration by parts yields

$$\int_a^b xf'(x)dx = [xf(x)]_a^b - \int_a^b f(x)dx$$

$$= bf(b) - af(a) - \int_a^b f(x)dx$$

A geometrical interpretation of this expression is that it corresponds to the shaded area in the above figure.

(Proof)

Let S be the surface of the shaded area, and assume that

$[a,b] = \bigcup_{i=1}^{n} [x_{i-1}, x_i], x_0 = a, x_n = b$.

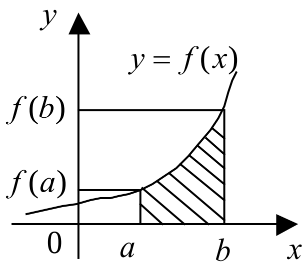
$$S = \sum_{i=1}^{n} \eta_i \{f(x_i) - f(x_{i-1})\} \qquad \left(^\exists \eta_i \in [x_i, x_{i-1}]\right)$$

$$= \sum_{i=1}^{n} \eta_i \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} (x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \eta_i f'(\eta_i')(x_i - x_{i-1}) \qquad \left(^\exists \eta_i' \in [x_i, x_{i-1}]\right)$$

$$\to \int_a^b xf'(x)dx \quad (n \to \infty)$$

（2） $\int_a^b xf(x)dx$ corresponds to the volume of the solid of revolution formed by rotating the shaded area (in the figure below) about the y-axis.



(Proof)

Let V be the volume to be obtained, and assume that

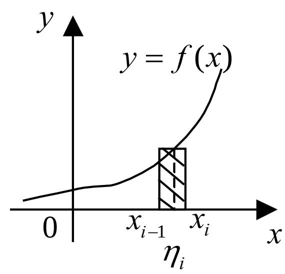
$[a,b]=\bigcup_{i=1}^{n}[x_{i-1},x_i], x_0=a, x_n=b$.

$$V=\sum_{i=1}^{n}\left\{\pi x_i^2 f(\eta_i)-\pi x_{i-1}^2 f(\eta_i)\right\} \qquad \left({}^{\exists}\eta_i\in[x_i,x_{i-1}]\right)$$

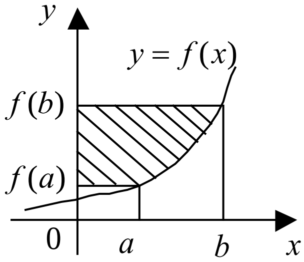
$$=\pi\sum_{i=1}^{n}\left(x_i^2-x_{i-1}^2\right)f(\eta_i)$$

$$=\pi\sum_{i=1}^{n}\left(x_i+x_{i-1}\right)f(\eta_i)\left(x_i-x_{i-1}\right)$$

$$\to 2\pi\int_a^b xf(x)dx \quad (n\to\infty)$$



(3) $\pi\int_a^b x^2 f'(x)dx$ corresponds to the volume of the solid of revolution formed by rotating the shaded area (in the figure on the right) about the y-axis.



(Proof)

Integration by parts yields

$$\pi\int_a^b x^2 f'(x)dx = \pi\left[x^2 f(x)\right]_a^b - 2\pi\int_a^b xf(x)dx$$

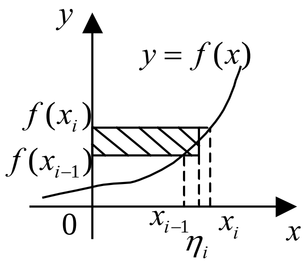
$$= \pi b^2 f(b) - \pi a^2 f(a) - 2\pi\int_a^b xf(x)dx$$

A geometrical interpretation is that this expression indicates the volume of the solid of revolution formed by rotating the shaded area about the y-axis.

(Proof)

Let V be the volume to be obtained, and assume that

$[a,b]=\bigcup_{i=1}^{n}[x_{i-1},x_i], x_0=a, x_n=b$.



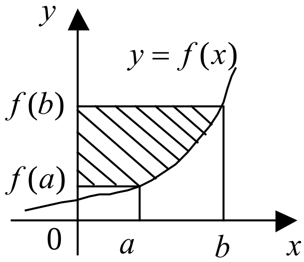
$$V=\sum_{i=1}^{n}\pi\eta_i^2\{f(x_i)-f(x_{i-1})\} \qquad \left(^\exists\eta_i\in[x_i,x_{i-1}]\right)$$

$$=\pi\sum_{i=1}^{n}\eta_i^2\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}(x_i-x_{i-1})$$

$$=\pi\sum_{i=1}^{n}\eta_i^2 f'(\eta_i')(x_i-x_{i-1}) \qquad \left(^\exists\eta_i'\in[x_i,x_{i-1}]\right)$$

$$\to \pi\int_a^b x^2 f'(x)dx \quad (n\to\infty)$$

（4） $2\pi\int_a^b xf'(x)f(x)dx$ corresponds to the volume of the solid of revolution formed by rotating the shaded area (in the figure on the right) about the x-axis.



（Proof）

Integration by parts yields

$$\pi\int_a^b f(x)^2 dx = \pi\left[xf(x)^2\right]_a^b - 2\pi\int_a^b xf'(x)f(x)dx$$

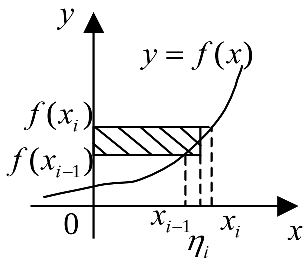
$$2\pi\int_a^b xf'(x)f(x)dx = \pi bf(b)^2 - \pi af(a)^2 - \pi\int_a^b f(x)^2 dx$$

A geometrical interpretation of this expression is that it represents the volume of the solid of revolution formed by rotating the shaded area about the x-axis.

（Proof）

Let V be the volume to be obtained, and assume that

$$[a,b] = \bigcup_{i=1}^{n}[x_{i-1}, x_i], x_0 = a, x_n = b.$$



$$V = \sum_{i=1}^{n}\left\{\pi\eta_i f(x_i)^2 - \pi\eta_i f(x_{i-1})^2\right\} \qquad \left({}^{\exists}\eta_i \in [x_i, x_{i-1}]\right)$$

$$= \pi\sum_{i=1}^{n}\eta_i\left\{f(x_i) + f(x_{i-1})\right\}\left\{f(x_i) - f(x_{i-1})\right\}$$

$$= \pi\sum_{i=1}^{n}\eta_i\left\{f(x_i) + f(x_{i-1})\right\}\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}(x_i - x_{i-1})$$

$$= \pi\sum_{i=1}^{n}\eta_i\left\{f(x_i) + f(x_{i-1})\right\}f'(\eta_i')(x_i - x_{i-1}) \qquad \left({}^{\exists}\eta_i' \in [x_i, x_{i-1}]\right)$$

$$\to 2\pi\int_a^b xf'(x)f(x)dx \quad (n \to \infty)$$

(5) $2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx$ corresponds to the surface of the solid of revolution formed by rotating the segment $[a,b]$ of $y=f(x)$ about the y-axis.

(Proof)

Let S be the area of the surface and assume that

$[a,b]=\bigcup_{i=1}^{n}[x_{i-1},x_i], x_0=a, x_n=b$.

$$S=\sum_{i=1}^{n}\pi(x_i+x_{i-1})\sqrt{(x_i-x_{i-1})^2+\{f(x_i)-f(x_{i-1})\}^2} \qquad ({}^{\exists}\eta_i\in[x_i,x_{i-1}])$$

$$=\pi\sum_{i=1}^{n}(x_i+x_{i-1})\sqrt{1+\left\{\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}\right\}^2}\times(x_i-x_{i-1})$$

$$=\pi\sum_{i=1}^{n}(x_i+x_{i-1})\sqrt{1+f'(\eta_i')^2}\times(x_i-x_{i-1}) \qquad ({}^{\exists}\eta_i'\in[x_i,x_{i-1}])$$

$$\to 2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx \qquad (n\to\infty)$$