

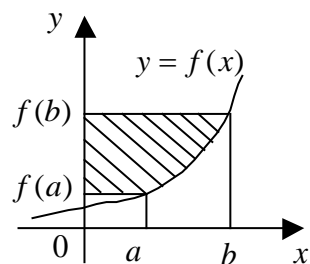
Let $y = f(x)$ be a continuous, monotonic function in $[a, b]$. Moreover, consider that $f'(x)$ exists and is continuous in that interval.

(1) $\int_a^b xf'(x)dx$ corresponds to the shaded area in the figure on the right.

(Proof)

Integration by parts yields

$$\begin{aligned} \int_a^b xf'(x)dx &= [xf(x)]_a^b - \int_a^b f(x)dx \\ &= bf(b) - af(a) - \int_a^b f(x)dx \end{aligned}$$



A geometrical interpretation of this expression is that it corresponds to the shaded area in the above figure.

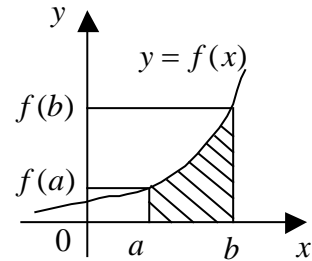
(Proof)

Let S be the surface of the shaded area, and assume that

$$[a, b] = \bigcup_{i=1}^n [x_{i-1}, x_i], x_0 = a, x_n = b.$$

$$\begin{aligned} S &= \sum_{i=1}^n \eta_i \{f(x_i) - f(x_{i-1})\} \quad (\exists \eta_i \in [x_i, x_{i-1}]) \\ &= \sum_{i=1}^n \eta_i \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} (x_i - x_{i-1}) \\ &= \sum_{i=1}^n \eta_i f'(\eta'_i) (x_i - x_{i-1}) \quad (\exists \eta'_i \in [x_i, x_{i-1}]) \\ &\rightarrow \int_a^b xf'(x)dx \quad (n \rightarrow \infty) \end{aligned}$$

(2) $\int_a^b xf(x)dx$ corresponds to the volume of the solid of revolution formed by rotating the shaded area (in the figure below) about the y -axis.



(Proof)

Let V be the volume to be obtained, and assume that

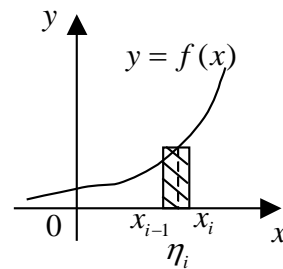
$$[a, b] = \bigcup_{i=1}^n [x_{i-1}, x_i], x_0 = a, x_n = b.$$

$$V = \sum_{i=1}^n \{ \pi x_i^2 f(\eta_i) - \pi x_{i-1}^2 f(\eta_i) \} \quad (\exists \eta_i \in [x_i, x_{i-1}])$$

$$= \pi \sum_{i=1}^n (x_i^2 - x_{i-1}^2) f(\eta_i)$$

$$= \pi \sum_{i=1}^n (x_i + x_{i-1}) f(\eta_i) (x_i - x_{i-1})$$

$$\rightarrow 2\pi \int_a^b xf(x)dx \quad (n \rightarrow \infty)$$

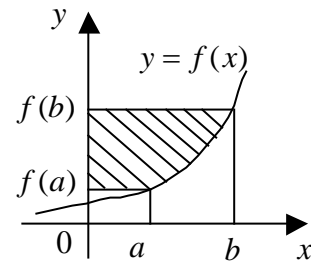


(3) $\pi \int_a^b x^2 f'(x) dx$ corresponds to the volume of the solid of revolution formed by rotating the shaded area (in the figure on the right) about the y -axis.

(Proof)

Integration by parts yields

$$\begin{aligned} \pi \int_a^b x^2 f'(x) dx &= \pi \left[x^2 f(x) \right]_a^b - 2\pi \int_a^b x f(x) dx \\ &= \pi b^2 f(b) - \pi a^2 f(a) - 2\pi \int_a^b x f(x) dx \end{aligned}$$



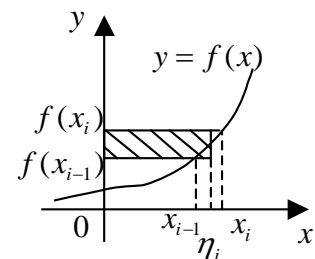
A geometrical interpretation is that this expression indicates the volume of the solid of revolution formed by rotating the shaded area about the y -axis.

(Proof)

Let V be the volume to be obtained, and assume that

$$[a, b] = \bigcup_{i=1}^n [x_{i-1}, x_i], \quad x_0 = a, \quad x_n = b.$$

$$\begin{aligned} V &= \sum_{i=1}^n \pi \eta_i^2 \{f(x_i) - f(x_{i-1})\} \quad (\exists \eta_i \in [x_i, x_{i-1}]) \\ &= \pi \sum_{i=1}^n \eta_i^2 \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} (x_i - x_{i-1}) \\ &= \pi \sum_{i=1}^n \eta_i^2 f'(\eta'_i) (x_i - x_{i-1}) \quad (\exists \eta'_i \in [x_i, x_{i-1}]) \\ &\rightarrow \pi \int_a^b x^2 f'(x) dx \quad (n \rightarrow \infty) \end{aligned}$$



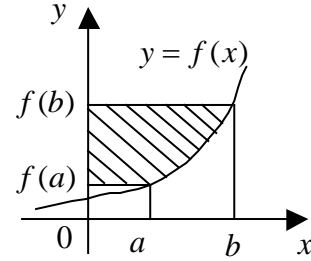
(4) $2\pi \int_a^b xf'(x)f(x)dx$ corresponds to the volume of the solid of revolution formed by rotating the shaded area (in the figure on the right) about the x -axis.

(Proof)

Integration by parts yields

$$\pi \int_a^b f(x)^2 dx = \pi [xf(x)^2]_a^b - 2\pi \int_a^b xf'(x)f(x)dx$$

$$2\pi \int_a^b xf'(x)f(x)dx = \pi bf(b)^2 - \pi af(a)^2 - \pi \int_a^b f(x)^2 dx$$



A geometrical interpretation of this expression is that it represents the volume of the solid of revolution formed by rotating the shaded area about the x -axis.

(Proof)

Let V be the volume to be obtained, and assume that

$$[a, b] = \bigcup_{i=1}^n [x_{i-1}, x_i], x_0 = a, x_n = b.$$

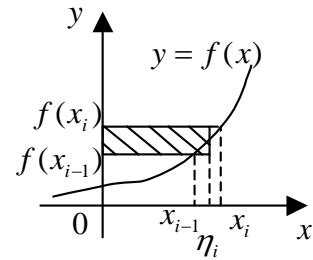
$$V = \sum_{i=1}^n \left\{ \pi \eta_i f(x_i)^2 - \pi \eta_i f(x_{i-1})^2 \right\} \quad (\exists \eta_i \in [x_i, x_{i-1}])$$

$$= \pi \sum_{i=1}^n \eta_i \{f(x_i) + f(x_{i-1})\} \{f(x_i) - f(x_{i-1})\}$$

$$= \pi \sum_{i=1}^n \eta_i \{f(x_i) + f(x_{i-1})\} \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} (x_i - x_{i-1})$$

$$= \pi \sum_{i=1}^n \eta_i \{f(x_i) + f(x_{i-1})\} f'(\eta'_i) (x_i - x_{i-1}) \quad (\exists \eta'_i \in [x_i, x_{i-1}])$$

$$\rightarrow 2\pi \int_a^b xf'(x)f(x)dx \quad (n \rightarrow \infty)$$



(5) $2\pi \int_a^b x \sqrt{1+f'(x)^2} dx$ corresponds to the surface of the solid of revolution formed by rotating the segment $[a, b]$ of $y = f(x)$ about the y -axis.

(Proof)

Let S be the area of the surface and assume that

$$[a, b] = \bigcup_{i=1}^n [x_{i-1}, x_i], x_0 = a, x_n = b.$$

$$S = \sum_{i=1}^n \pi (x_i + x_{i-1}) \sqrt{(x_i - x_{i-1})^2 + \{f(x_i) - f(x_{i-1})\}^2} \quad (\exists \eta_i \in [x_i, x_{i-1}])$$

$$= \pi \sum_{i=1}^n (x_i + x_{i-1}) \sqrt{1 + \left\{ \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right\}^2} \times (x_i - x_{i-1})$$

$$= \pi \sum_{i=1}^n (x_i + x_{i-1}) \sqrt{1 + f'(\eta_i)^2} \times (x_i - x_{i-1}) \quad (\exists \eta_i' \in [x_i, x_{i-1}])$$

$$\rightarrow 2\pi \int_a^b x \sqrt{1 + f'(x)^2} dx \quad (n \rightarrow \infty)$$