The series expansion of B-function ($p, q > 0$ and not integers) is:

$$B(p,q) = \frac{1}{2^{p+q-1}}\left(\frac{1}{p}\sum_{n=0}^{\infty} a_n + \frac{1}{q}\sum_{n=0}^{\infty} b_n\right)$$

$$a_0 = 1$$

$$b_0 = 1$$

$$a_n = \frac{(q-1)(q-2)\cdots(q-n)}{(p+1)(p+2)\cdots(p+n)} = \prod_{i=1}^{n}\frac{q-i}{p+i}$$

$$b_n = \frac{(p-1)(p-2)\cdots(p-n)}{(q+1)(q+2)\cdots(q+n)} = \prod_{i=1}^{n}\frac{p-i}{q+i}$$

$$(i = 1,2,3,\cdots)$$

(Proof)

$$\int_0^{\frac{1}{2}} x^{p-1}(1-x)^{q-1}dx = \left[\frac{1}{p}x^p(1-x)^{q-1}\right]_0^{\frac{1}{2}} + \frac{q-1}{p}\int_0^{\frac{1}{2}} x^p(1-x)^{q-2}dx$$

$$= \frac{1}{2^{p+q-1}}\frac{1}{p} + \left[\frac{q-1}{p(p+1)}x^{p+1}(1-x)^{q-2}\right]_0^{\frac{1}{2}} + \frac{(q-1)(q-2)}{p(p+1)}\int_0^{\frac{1}{2}} x^{p+1}(1-x)^{q-3}dx$$

$$= \frac{1}{2^{p+q-1}}\frac{1}{p} + \frac{1}{2^{p+q-1}}\frac{q-1}{p(p+1)} + \frac{(q-1)(q-2)}{p(p+1)}\int_0^{\frac{1}{2}} x^{p+1}(1-x)^{q-3}dx$$

If we continue the above and define the following integer sequence $\{a_n\}(n = 0,1,2,\cdots)$:

$$a_0 = 1$$

$$a_n = \frac{(q-1)(q-2)\cdots(q-n)}{(p+1)(p+2)\cdots(p+n)} = \prod_{i=1}^{n}\frac{q-i}{p+i}$$

$$(n = 1,2,3,\cdots)$$

,

we obtain:

$$\int_0^{\frac{1}{2}} x^{p-1}(1-x)^{q-1}dx = \frac{1}{2^{p+q-1}}\frac{1}{p}\left\{\sum_{i=0}^{n} a_i + (q-n-1)a_n\int_0^{\frac{1}{2}} x^{p+n}(1-x)^{q-n-2}dx\right\}.$$

Next, let's show that

$$\frac{q-n-1}{p}a_n\int_0^{\frac{1}{2}} x^{p+n}(1-x)^{q-n-2}dx \to 0 \qquad (n \to \infty).$$

Consider that

$$\frac{x}{1-x}=t$$.

Then, we have:

$$x=\frac{t}{1+t}$$

$$1-x=1-\frac{t}{1+t}$$

$$=\frac{1}{1+t}$$

$$\frac{dx}{dt}=\frac{d}{dt}\left(\frac{t}{1+t}\right)$$

$$=\frac{d}{dt}\left(1-\frac{1}{1+t}\right)$$

$$=\frac{1}{(1+t)^2}$$

$$\left|\frac{q-n-1}{p}\int_0^{\frac{1}{2}} x^{p+n}(1-x)^{q-n-2}dx\right|=\left|\frac{q-n-1}{p}\int_0^{\frac{1}{2}} x^{p-1}(1-x)^{q-1}\left(\frac{x}{1-x}\right)^{n+1}dx\right|$$

$$=\left|\frac{q-n-1}{p}\int_0^1\left(\frac{t}{1+t}\right)^{p-1}\left(\frac{1}{1+t}\right)^{q-1}t^{n+1}\frac{1}{(1+t)^2}dt\right|$$

$$=\left|\frac{q-n-1}{p}\int_0^1\frac{t^{p+n+1}}{(1+t)^{p+q}}dt\right|$$

$$<\left|\frac{q-n-1}{p}\int_0^1 t^{p+n+1}dt\right|$$

$$=\left|\frac{q-n-1}{p(p+n+2)}\right|\to\frac{1}{p}\qquad(n\to\infty)$$

Here, we assume that n is sufficiently large.

An integer I that satisfies $q<I$ exists, given that $1<I$.

If $q<I$, then $0<1-\frac{p+q}{p+i}<1$

$$|a_n|=\left|\prod_{i=1}^{n}\frac{q-i}{p+i}\right|$$

$$=\left|\prod_{i=1}^{I-1}\frac{q-i}{p+i}\right|\prod_{i=I}^{n}\frac{i-q}{i+p}$$

$$= \left| \prod_{i=1}^{I-1} \frac{q-i}{p+i} \right| \exp\left(\log \prod_{i=I}^{n} \frac{i-q}{i+p} \right)$$

$$= \left| \prod_{i=1}^{I-1} \frac{q-i}{p+i} \right| \exp\left(\sum_{i=I}^{n} \log \frac{i-q}{i+p} \right)$$

$$\sum_{i=I}^{n} \log \frac{i-q}{i+p} = \sum_{i=I}^{n} \log\left(1 - \frac{p+q}{i+p} \right)$$

$$= \sum_{i=I}^{n} \left\{ -\left(\frac{p+q}{i+p} \right) - \frac{1}{2}\left(\frac{p+q}{i+p} \right)^2 - \frac{1}{3}\left(\frac{p+q}{i+p} \right)^3 - \cdots \right\} \to -\infty \quad (n \to \infty)$$

Therefore,

$$|a_n| \to 0 \qquad (n \to \infty)$$

Then, we have:

$$\frac{q-n-1}{p} a_n \int_0^{\frac{1}{2}} x^{p+n} (1-x)^{q-n-2} dx \to 0 \qquad (n \to \infty)$$

Using a similar method, we construct the integer sequence $\{b_n\} (n = 0,1,2,\cdots)$ from $\int_{\frac{1}{2}}^{1} x^{p-1} (1-x)^{q-1} dx$, and then we can prove that the sequence $\{b_n\}$ converges.

(End of proof)