Addition, subtraction, multiplication, and division of algebraic equations

I hit upon the following method of calculation during my sophomore year of college when I was working on the representation of algebraic equations. When I taught this method to first-year students at a senior high school where I was working as a trainee teacher during my senior year of college, they were impressed, and today, this method appears to be widely used.

Let us carry out the following calculations by using this method.

$(X - 3)(2X + 1)(X^2 + 3X + 1) = ?$
$(X^6 + 4X^5 - X^4 - 5X^3 + 2X^2 - 3X - 2)/(X^2 + 2X - 2) = ?$

Solution

		1	−3
X		2	1
		1	−3
	2	−6	
	2	−5	−3

			2	−5	−3
X			1	3	1
			2	−5	−3
		6	−15	9	
	2	−5	−3		
	2	1	−16	4	−3

			1	2	−3	5	−14		
1	2	−2)	1	4	−1	−5	2	−3	−2
			1	2	−2				
				2	1				
				2	4	−4			
					−3	−1			
					−3	−6	6		
						5	−4		
						5	10	−10	
							−14	7	
							−14	−28	28
								35	−30

Therefore,

$(X - 3)(2X + 1)(X^2 + 3X + 1) = 2X^4 + X^3 - 16X^2 + 4X - 3$

$(X^6 + 4X^5 - X^4 - 5X^3 + 2X^2 - 3X - 2)/(X^2 + 2X - 2) = X^4 + 2X^3 - 3X^2 + 5X - 14$, and the remainder is $(35X - 30)$

This method can be interpreted as one that involves carry-free addition, subtraction, multiplication, and division.

Let's consider the recurring decimal. For example, for calculating 3/11, we have:

```
          0.   2    7    2
        ____________________
  1 1  )  3    0
          2    2
        ____________
               8    0
               7    7
             ___________
                    3    0
                    2    2
                  __________
                         8    0
```

,

which results in 3/11 = 0.27272727…

We can use the above to calculate algebraic equations. For example, if we calculate $\frac{1}{(x+1)(x+3)}$, we obtain:

```
                 0.    0    1    -4    13   -40
               _____________________________________
  1    4    3  )  1.    0    0
                  1     4    3
               _________________
                       -4   -3     0
                       -4  -16   -12
                     __________________
                             13    12     0
                             13    52    39
                           ________________
                                  -40   -39     0
                                  -40  -160  -120
                                ___________________
                                        121   120
```

,

and then

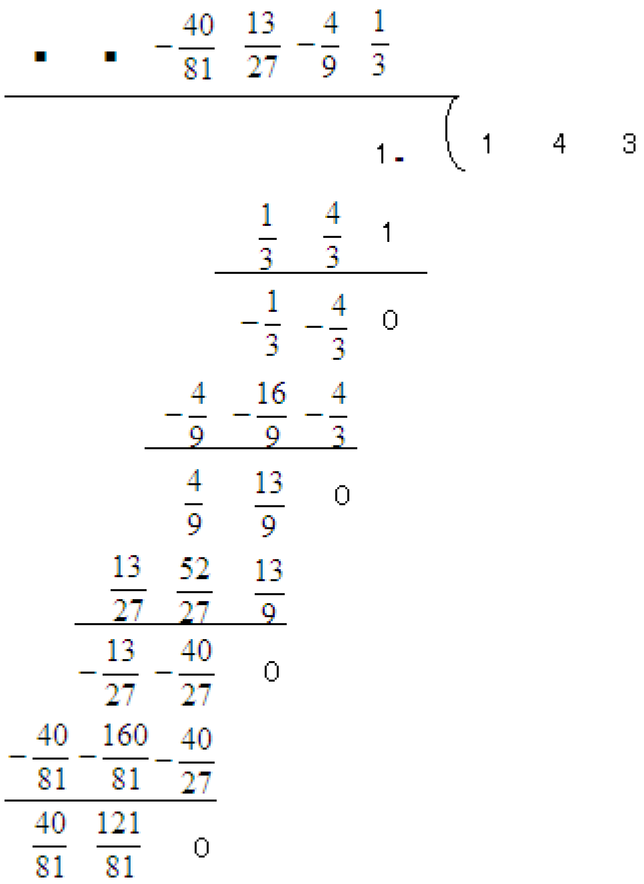
$$\frac{1}{(x+1)(x+3)} = \frac{1}{x^2} - \frac{4}{x^3} + \frac{13}{x^4} - \frac{40}{x^5} + \cdots$$

This result coincides with the Laurent expansion taught in undergraduate courses.

Here, even though we are not looking for the convergence of x, it converges for $|x| > 3$.

Further investigation may result in new applications.

The division method is changed and executed leftwards.



$$\frac{1}{(x+1)(x+3)}=\frac{1}{3}-\frac{4}{9}x+\frac{13}{27}x^2-\frac{40}{82}x^3+\cdots$$

It matches the results of the Taylor expansion taught at high school.

Here, we are not looking for the region of convergence of x, but note that it converges for $|x|<1$.

The two calculations above resemble a expansion expansion that uses only addition, subtraction, multiplication and division, without the derivative of the Taylor expansion and the complex integral of the Laurent expansion.

Let us try to generalize it. Consider the simple rational function:

$$f(x)=\frac{a_m x^m + a_{m-1}x^{m-1}+\cdot\cdot+a_1 x+a_0}{b_n x^n + b_{n-1}x^{n-1}+\cdot\cdot+b_1 x+b_0}=\frac{A(x)}{B(x)}$$

Here, the numerator and denominator are predefined, with $b_n \neq 0$ and $b_0 \neq 0$.

The calculations so far considered $x=0$, and generalization implies finding the expansion centered on $x=\alpha$. This generalization is transformed into a division between the terms $(a_n, a'_{n-1}, \cdot\cdot, a'_0)(b_m, b'_{m-1}, \cdot\cdot, b'_0)$, and in this case it is sufficient to consider $x=0$.

$$f(x)=\frac{a_m x^m + a_{m-1}x^{m-1}+\cdot\cdot+a_1 x+a_0}{b_n x^n + b_{n-1}x^{n-1}+\cdot\cdot+b_1 x+b_0}=\frac{a_m(x-\alpha)^m + a'_{m-1}(x-\alpha)^{m-1}+\cdot\cdot+a'_1(x-\alpha)+a'_0}{b_n(x-\alpha)^n + b'_{n-1}(x-\alpha)^{n-1}+\cdot\cdot+b'_1(x-\alpha)+b'_0}$$

For $m>n$, it is possible to perform the following transformation:

$$f(x)=\frac{a_m x^m + a_{m-1}x^{m-1}+\cdot\cdot+a_1 x+a_0}{b_n x^n + b_{n-1}x^{n-1}+\cdot\cdot+b_1 x+b_0}$$

$$=\frac{a_m}{b_n}x^{m-n}+a''_{m-n-1}x^{m-n-1}+\cdot\cdot+a''_{n+1}x^{n+1}+\frac{a''_n x^n + a''_{n-1}x^{n-1}+\cdot\cdot+a''_1 x+a''_0}{b_n x^n + b_{n-1}x^{n-1}+\cdot\cdot+b_1 x+b_0}$$

To verify the possibility of performing series expansion, it is sufficient to consider just the series expansion of

$$\frac{a''_n x^n + a''_{n-1}x^{n-1}+\cdot\cdot+a''_1 x+a''_0}{b_n x^n + b_{n-1}x^{n-1}+\cdot\cdot+b_1 x+b_0}$$

The proof for $m=n$ is sufficient.

For $m<n$, the equation can be reconstructed considering $a_n, a_{n-1}, \cdot\cdot, a_{m+1}=0$, and the proof for the case of $m=n$ is sufficient.

$$c_0=\frac{a_0}{b_0}$$

$$d_{1,i}=c_0 b_i (1 \le i \le n)$$

$$a_{1,i}=a_i-d_{1,i}(1 \le i \le n)$$

				c_0				
a_n	$\cdots$	a_2	a_1	a_0	(b_n	$\cdots$	b_1	b_0
$d_{1,n}$	$\cdots$	$d_{1,2}$	$d_{1,1}$	a_0				
$a_{1,n}$	$\cdots$	$a_{1,2}$	$a_{1,1}$	0				

As a result of this calculation, we obtain:

$$A(x)=c_0 B(x)+a_{1,n}x^n+\cdots+a_{1,1}x$$

$$c_1 = \frac{a_{1,1}}{b_0} = \frac{a_1 - c_0 b_1}{b_0} = \frac{a_1 b_0 - a_0 b_1}{{b_0}^2}$$

$$\begin{array}{cccccc|cccc}
 & & & & c_1 & c_0 & & & & \\
\hline
 & a_n & \cdots & a_2 & a_1 & a_0 & b_n & \cdots & b_1 & b_0 \\
 & d_{1,n} & \cdots & d_{1,2} & d_{1,1} & a_0 & & & & \\
\hline
 & a_{1,n} & \cdots & a_{1,2} & a_{1,1} & 0 & & & & \\
d_{2,n} & d_{2,n-1} & \cdots & d_{2,1} & a_{1,1} & & & & & \\
\hline
a_{2,n} & \cdots & a_{2,2} & a_{2,1} & 0 & & & & &
\end{array}$$

$$d_{2,i} = c_1 b_i (1 \le i \le n)$$

$$a_{2,i} = a_{1,i+1} - d_{2,i} (1 \le i < n)$$

$$a_{2,n} = -d_{2,n}$$

As a result of this calculation, we obtain:

$$A(x) = (c_0 + c_1 x) B(x) + a_{2,n} x^{n+1} + \cdots + a_{2,1} x^2$$

$$\begin{array}{ccccccc|cccc}
 & & & & c_2 & c_1 & c_0 & & & & \\
\hline
 & & a_n & \cdots & a_2 & a_1 & a_0 & b_n & \cdots & b_1 & b_0 \\
 & & d_{1,n} & \cdots & d_{1,2} & d_{1,1} & a_0 & & & & \\
\hline
 & & a_{1,n} & \cdots & a_{1,2} & a_{1,1} & 0 & & & & \\
 & d_{2,n} & d_{2,n-1} & \cdots & d_{2,1} & a_{1,1} & & & & & \\
\hline
 & a_{2,n} & \cdots & a_{2,2} & a_{2,1} & 0 & & & & & \\
d_{3,n} & d_{3,n-1} & \cdots & d_{3,1} & a_{2,1} & & & & & & \\
\hline
a_{3,n} & \cdots & a_{3,2} & a_{3,1} & 0 & & & & & &
\end{array}$$

Continuing this calculation yields, for $k > 1$:

$$A(x) = (c_0 + c_1 x + \cdots + c_k x^k) B(x) + a_{k+1,n} x^{n+k} + \cdots + a_{k+1,1} x^{k+1}.$$

generalizing x to complex numbers and considering that no x exists satisfying $B(x) \neq 0$, we have:

$$\frac{A(x)}{B(x)} = c_0 + c_1 x + \cdots + c_k x^k + \frac{a_{k+1,n} x^{n+k} + \cdots + a_{k+1,1} x^{k+1}}{B(x)} \quad \cdot \cdot \cdot \cdot ①$$

Consider the n solutions that satisfy $B(x) = 0$ as $\beta_1, \beta_2, \cdots, \beta_n$, and

$r = \min_{1 \le i \le n} |\beta_i|$.

If $|x| < r$, $\frac{A(x)}{B(x)}$ is differentiable and can be expanded in a Taylor expansion, and an $\xi(|\xi| < r)$ exists satisfying

$$f(x) = \frac{A(x)}{B(x)} = f(0) + f^{(1)}(0)x + \cdots + \frac{f^{(k-1)}(0)}{(k-1)!}x^{k-1} + \frac{f^{(k)}(\xi)}{k!}x^k \cdot \cdot \cdot \cdot \cdot ②.$$

Here, we attempt to prove that ① is equivalent to ② by mathematical induction:

$$f(0) = \frac{A(0)}{B(0)} = \frac{a_0}{b_0} = c_0$$

$$f^{(1)}(0) = \frac{A^{(1)}(0)B(0) - A(0)B^{(1)}(0)}{B(0)^2} = \frac{a_1 b_0 - a_0 b_1}{b_0{}^2} = c_1$$

Therefore, ① is equivalent to ② for $i = 0,1$.

Suppose that ① is equivalent to ② for $k = i$.

$$f(x) = \frac{A(x)}{B(x)} = c_0 + c_1 x + \cdots + c_i x^i + \frac{a_{i+1,n}x^{n+i} + \cdots + a_{i+1,1}x^{i+1}}{B(x)}$$

$$f^{(i+1)}(x) = \left(c_0 + c_1 x + \cdots + c_i x^i + \frac{a_{i+1,n}x^{n+i} + \cdots + a_{i+1,1}x^{i+1}}{B(x)} \right)^{(i+1)}$$

$$= \left(x^{i+1} \frac{a_{i+1,n}x^{n-1} + \cdots + a_{i+1,1}}{B(x)} \right)^{(i+1)}$$

$$= (i+1)!\frac{a_{i+1,n}x^{n-1} + \cdots + a_{i+1,1}}{B(x)} + {}_{i+1}C_1 (i+1)!\, x \left(\frac{a_{i+1,n}x^{n-1} + \cdots + a_{i+1,1}}{B(x)} \right)^{(1)}$$

$$+ \cdots + x^{i+1} \left(\frac{a_{i+1,n}x^{n-1} + \cdots + a_{i+1,1}}{B(x)} \right)^{(i+1)}$$

Therefore, we have:

$$\frac{f^{(i+1)}(0)}{(i+1)!} = \frac{a_{i+1,1}}{b_0}$$

This equation is equivalent to a division between algebraic equations. In other words, for $k = i+1$, ① is equivalent to ②.

Next, the calculation below is carried out rightwards.

$$
\begin{array}{lllllllll}
 & & & & c'_0 \bullet & c'_{-1} & c'_{-2} & & \\
\cline{2-7}
b_n \cdot\cdot\ b_1\ b_0 \Big) & a_n & a_{n-1} & \cdots & a_0 & & & & \\
 & a_n & d'_{1,n-1} & \cdots & d'_{1,0} \bullet & & & & \\
\cline{2-6}
 & 0 & a'_{1,n-1} & a'_{1,n-2} & \cdots & 0 & & & \\
 & & a'_{1,n-1} & d'_{2,n-2} & \cdots & d'_{2,-1} & & & \\
\cline{3-7}
 & & 0 & a'_{2,n-2} & a'_{2,n-3} & \cdots & 0 & & \\
 & & & a'_{2,n-2} & d'_{3,n-3} & \cdots & d'_{3,-2} & & \\
\cline{4-8}
 & & & 0 & a'_{3,n-3} & a'_{3,n-4} & \cdots & 0 &
\end{array}
$$

$$A(x)=c'_0B(x)+\frac{a'_{1,n-1}}{x^{1-n}}+\cdots+a'_{1,0}$$

$$A(x)=\left(c'_0+\frac{c'_{-1}}{x}\right)B(x)+\frac{a'_{2,n-2}}{x^{2-n}}+\cdots+\frac{a'_{2,-1}}{x}$$

Considering $k>1$, we have:

$$A(x)=\left(c'_0+\frac{c'_{-1}}{x}+\cdots+\frac{c'_{-k}}{x^k}\right)B(x)+\frac{a'_{k+1,n-k-1}}{x^{k+1-n}}+\cdots+\frac{a'_{k+1,-k}}{x^k}$$

$$R=\max_{1\le i\le n}|\beta_i|.$$

If $|x|>R$, since $B(x)\neq 0$, we have:

$$\frac{A(x)}{B(x)}=c'_0+\frac{c'_{-1}}{x}+\cdots+\frac{c'_{-k}}{x^k}+\frac{\dfrac{a'_{k+1,n-k-1}}{x^{k+1-n}}+\cdots+\dfrac{a'_{k+1,-k}}{x^k}}{B(x)}\quad\cdot\cdot\cdot\cdot ③$$

If $|x|>R$, then $\dfrac{A(x)}{B(x)}$ is a regular function that can be expanded into a Laurent expansion. In addition, ③ must coincide with the Laurent expansion.

Let $x=\dfrac{1}{X}$.

If $|X|<\dfrac{1}{R}$, $A\left(\dfrac{1}{X}\right)\Big/B\left(\dfrac{1}{X}\right)$ is differentiable and can be expanded into a Taylor expansion. The

Taylor expansion of $A\left(\frac{1}{X}\right)\Big/B\left(\frac{1}{X}\right)$ coincides with the division $A\left(\frac{1}{X}\right)\Big/B\left(\frac{1}{X}\right)$ leftwards.

Reverting using $x=\frac{1}{X}$ should coincide with the Laurent expansion.

Considering

$$A'(X)=X^{n}A\left(\frac{1}{X}\right)$$

$$B'(X)=X^{n}B\left(\frac{1}{X}\right)$$

we calculate

$$\frac{A'(X)}{B'(X)}=\frac{a_n+\cdot\cdot+a_1X^{n-1}+a_0X^{n}}{b_n+\cdot\cdot+b_1X^{n-1}+b_0X^{n}}$$

to the leftward direction.

$$\begin{array}{cccccccl}
 & & & c'_{-2} & c'_{-1} & c'_{0} & & \\
\cline{3-6}
 & & a_0 & \cdots & a_{n-1} & a_n & \big(& b_0\ b_1\ \cdot\cdot\ b_n \\
 & & d'_{1,0} & \cdots & d'_{1,n-1} & a_n & & \\
\cline{3-6}
 & & a'_{1,0} & \cdots & a'_{1,n-1} & 0 & & \\
 & d'_{2,-1} & \cdots & d'_{2,n-2} & a'_{1,n-1} & & & \\
\cline{2-5}
 & a'_{2,-1} & \cdots & a'_{2,n-2} & 0 & & & \\
d'_{3,-2} & \cdots & d'_{3,n-3} & a'_{2,n-2} & & & & \\
\cline{1-4}
a'_{3,-2} & \cdots & a'_{3,n-3} & 0 & & & &
\end{array}$$

$$A'(X)=c'_0B'(X)+a'_{1,0}X^{n}+\cdots+a'_{1,n-1}X$$

$$A'(X)=(c'_0+c'_{-1}X)B'(X)+a'_{2,-1}X^{n+1}+\cdots+a'_{2,n-2}X^{2}$$

Considering $k>1$,

$$A'(X)=(c'_0+c'_{-1}X+\cdot\cdot+c'_{-k}X^{k})B'(X)+a'_{k+1.-k}X^{n+k}+\cdots+a'_{k+1.n-k-1}X^{k+1}$$

$$X^{n}A\left(\frac{1}{X}\right)=(c'_0+c'_{-1}X+\cdot\cdot+c'_{-k}X^{k})X^{n}B\left(\frac{1}{X}\right)+a'_{k+1.-k}X^{n+k}+\cdots+a'_{k+1.n-k-1}X^{k+1}$$

$$A\left(\frac{1}{X}\right)=(c'_0+c'_{-1}X+\cdot\cdot+c'_{-k}X^{k})B\left(\frac{1}{X}\right)+a'_{k+1.-k}X^{k}+\cdots+a'_{k+1.n-k-1}X^{k+1-n}$$

Reverting $x=\frac{1}{X}$, it becomes

$$\frac{A(x)}{B(x)}=c'_0+\frac{c'_{-1}}{x}+\cdot\cdot+\frac{c'_{-k}}{x^{k}}+\frac{\frac{a'_{k+1,-k}}{x^{k}}+\cdots+\frac{a'_{k+1,n-k-1}}{x^{k+1-n}}}{B(x)},$$

which coincides with the Laurent expansion.